

On the Complexity of Barrier Resilience for Fat Regions

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Abstract

In the *barrier resilience* problem (introduced Kumar *et al.*, Wireless Networks 2007), we are given a collection of regions of the plane, acting as obstacles, and we would like to remove the minimum number of regions so that two fixed points can be connected without crossing any region. In this paper, we show that the problem is NP-hard when the regions are fat (even when they are axis-aligned rectangles of aspect ratio $1 : (1 + \varepsilon)$). We also show that the problem is fixed-parameter tractable (FPT) for such regions. Using our FPT algorithm, we show that if the regions are β -fat and their arrangement has bounded ply Δ , there is a $(1 + \varepsilon)$ -approximation that runs in $O(2^{f(\Delta, \varepsilon, \beta)} n^7)$ time, where $f \in O(\frac{\Delta^2 \beta^6}{\varepsilon^4} \log(\beta \Delta / \varepsilon))$.

1 Introduction

The *barrier resilience* problem asks for the minimum number of spatial regions from a collection \mathcal{D} that need to be removed, such that two given points p and q are in the same connected component of the complement of the union of the remaining regions. This problem was posed originally in 2005 by Kumar *et al.* [10, 11], motivated from sensor networks. In their formulation, the regions are unit disks (sensors) in some rectangular strip $B \subset \mathbb{R}^2$, where each sensor is able to detect movement inside its disk. The question is then how many sensors need to fail before an entity can move undetected from one side of the strip to the other (that is, how *resilient* to failure the sensor system is). Kumar *et al.* present a polynomial time algorithm to compute the resilience in this case. They also consider the case where the regions are disks in an annulus, but their approach cannot be used in that setting.

1.1 Related Work

Despite the seemingly small change from a rectangular strip to an annulus, the second problem still remains open, even for the case in which regions are unit disks in \mathbb{R}^2 . There has been partial progress towards settling the question: Bereg and Kirkpatrick [2] present a factor 5/3 approximation algorithm for the unit disk case. Afterwards, Alt *et al.* [1] and Tseng and Kirkpatrick [14] independently show that if the regions are line segments in \mathbb{R}^2 , the problem is NP-hard.

The problem of covering barriers with sensors is very current and has received a lot of attention in the sensor network community (e.g. [3, 4, 8]). In the algorithms community, problems involving

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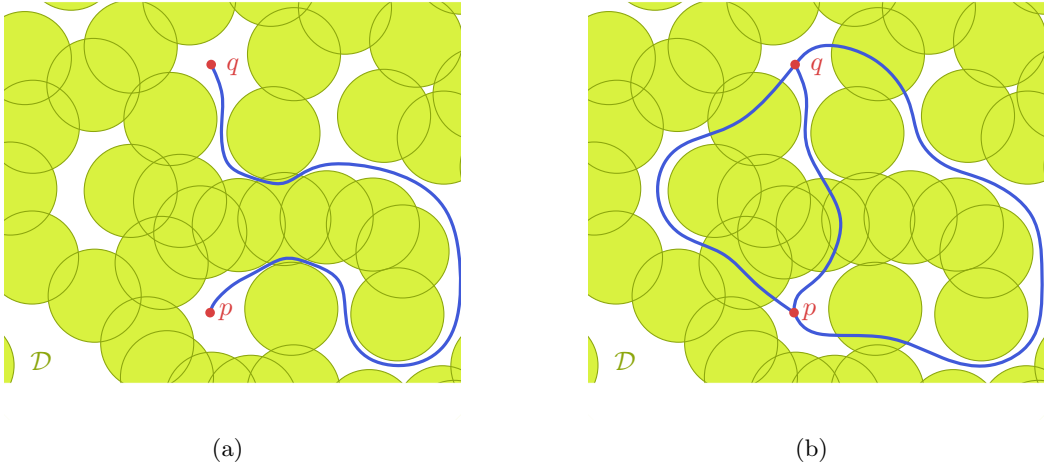


Figure 1: (a) A path of minimum resilience (equal to 1). (b) Three paths with minimum thickness (2).

region intersection graphs are also popular. Gibson *et al.* [7] study the opposite problem of ours: compute the maximum number of disks one can remove such that p and q are still separated.

1.2 Results

The NP-hardness constructions in [1] and [14] both rely on the fact that line segments are long and skinny. In this paper, we show that it is not the skinniness of line segments what makes the problem hard. We show that when the regions are *fat* regions [5] in \mathbb{R}^2 , the problem is still NP-hard, even for the case in which the regions are axis-aligned rectangles of aspect ratio $1 : 1 + \varepsilon$ and $1 + \varepsilon : 1$. Our construction suggests that hardness relies on the fact that the regions can cross each other, and it does not easily extend to the case where the regions are disks. In Section 3 we discuss this in more depth.

To complement our hardness result, we show in Section 4 that when the regions are fat the problem is fixed-parameter tractable on the resilience. In Section 5 we show that, if the system of regions has bounded *ply* [13], we can use the FPT result to obtain an approximation scheme. In particular, the constructive results apply to the original unit disk coverage setting (since unit disks are fat regions). Our results rely on tools and techniques from both computational geometry and graph theory.

2 Preliminaries

We denote with p and q the points that need to be connected, and with \mathcal{D} the set of regions that represent the sensors. We say that any collection of objects in the plane are *pseudodisks* if the boundaries of any two of them intersect transversely at most twice.

We formally define the concepts of *resilience* and *thickness* introduced in [2]. The *resilience* of a path π between two points p and q , denoted $r(\pi)$, is the number of regions of \mathcal{D} intersected by π . Given two points p and q , the *resilience* of p and q , denoted $r(p, q)$, is the minimum resilience over all paths connecting p and q . In other words, the resilience between p and q is the minimum number of regions of \mathcal{D} that need to be removed to have a path between p and q that does not intersect any region. From now on, we assume that neither p nor q are contained in any region of

\mathcal{D} . Notice that such a region must always be counted in the minimum resilience paths, hence we can ignore them (and update the resilience we obtain accordingly).

Often it will be useful to refer to the arrangement induced by the regions of \mathcal{D} , which we denote by $\mathcal{A}(\mathcal{D})$. Based on this arrangement we define a weighted graph $G_{\mathcal{A}(\mathcal{D})}$ as follows: each vertex of $G_{\mathcal{A}(\mathcal{D})}$ corresponds to a cell of $\mathcal{A}(\mathcal{D})$. We also add directed edges between any pair of neighboring cells. An edge has cost 1 if following the edge we enter an element of \mathcal{D} , or cost 0 if we are leaving it instead.

The *thickness* of a path π between p and q , denoted $t(\pi)$, equals the number of sensor region intersections of π (possibly counting the same region multiple times). Given two points p and q , the *thickness of p and q* , denoted $t(p, q)$, is the value $|\mathfrak{P}_{G_{\mathcal{A}(\mathcal{D})}}(p, q)| + \Delta(p)$, where $\mathfrak{P}_{G_{\mathcal{A}(\mathcal{D})}}(p, q)$ is a shortest path in $G_{\mathcal{A}(\mathcal{D})}$ from the cell of p to the cell of q , and $\Delta(p)$ equals the number of regions that contain p . Note that the thickness between two points can be efficiently computed in polynomial time. Also notice that the resilience (or thickness) between only two points only depends on the cells to which the points belong to. Hence, we can naturally extend the definitions of thickness to encompass two cells of $\mathcal{A}(\mathcal{D})$, or a cell of $\mathcal{A}(\mathcal{D})$ and a point $p \in \mathbb{R}^2$.

Note that thickness and resilience can be different (since entering the same region several times has no impact in the resilience, but is counted every time for the thickness), see Fig. 1. However, as we will see later, the thickness (and the associated shortest path) will help us find a path of low resilience.

The following useful lemma follows directly from Lemma 1 in [2], and applies to the case where the regions in \mathcal{D} are unit disks. In the following, “well-separated” is in the sense used in [2] (i.e., the distance between p and q is at least $2\sqrt{3}$).[□]

Lemma 1. *Let \mathcal{D} be a set of unit disks, and let $S \subset \mathcal{D}$ be an optimal solution. If p, q are well-separated, then no disk of S appears more than twice.*

Corollary 1. *When the regions of \mathcal{D} are unit disks, the thickness between two well-separated points is at most twice their resilience.*

3 NP-hardness

In this section we will show that computing the resilience of certain types of fat regions is NP-hard. First we show NP-hardness for general connected regions, and later we extend it to axis-aligned rectangles of aspect ratio $1 : 1 + \varepsilon$ and $1 + \varepsilon : 1$. We start the section establishing some useful graph-theoretical results.

3.1 Maximum independent set and Euler paths in oddly embeddable graphs

Let G be a graph, and let p be a point in the plane. Let Γ be an embedding of G into the plane, which behaves properly (vertices go to distinct points, edges are curves that do not meet vertices other than their endpoints and do not triple cross), and such that p is not on a vertex or edge of the embedding. We say Γ is an *odd* embedding around p if it has the following property: every cycle of G has odd length if and only if the winding number of the corresponding closed curve in the plane in Γ around p is odd. Fig. 2 shows some examples. We say a graph G is *oddly embeddable* if there exists an odd embedding Γ for it.

[□]Note that the well-separatedness of p and q is only needed to prove a factor 2 rather than 3. Everything still works for ill-separated points, at a slight increase of the constants. Our most general theorem statements, for β -fat regions, do not require the points to be well-separated.

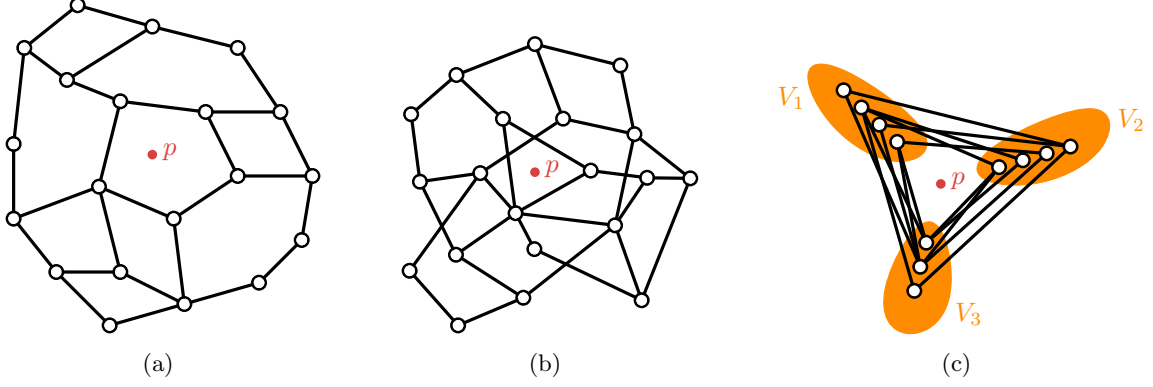


Figure 2: (a) A planar odd embedded graph. (b) A non-planar one. (c) A tripartite graph, oddly embedded around p .

Lemma 2. *Every tripartite graph is oddly embeddable.*

Proof. The vertices of a tripartite graph G can be divided into three groups V_1, V_2, V_3 such that there are no internal edges in any of these groups. Now, consider a triangle Δ around p . We create a drawing Γ where all vertices in V_1 are close to one corner of Δ , the vertices in V_2 are close to a second corner, and the vertices in V_3 are close to the remaining corner. All edges are straight line segments. See Fig. 2(c).

Consider the graph H obtained from G by contracting all vertices in V_i to a single vertex v_i ; H is a triangle (or a subgraph of a triangle). Now consider any cycle in G , and project it to H . Since there were no edges in G connecting vertices within a group V_i , this does not change the length of the cycle, nor does it change the winding number around p . Any two consecutive edges from v_i to v_j , and back from v_j to v_i , do not influence the parity of the length of the cycle, nor the winding number around p , so we can remove them from the cycle. We are left with a cycle of length $3w$ and winding number w or $-w$, for some integer w . Clearly, $3w$ is odd if and only if w is odd. Therefore, Γ is an odd embedding of G , as required. \square

The *maximum independent set* problem in a graph asks for the largest set of vertices in the graph such that no two vertices in the set are connected by an edge. This problem is well-known to be NP-hard on general graphs. In fact, it remains NP-hard for tripartite graphs. A simple proof is included for completeness, and because we need the argument later.

Observation 1. *Let $G = (V, E)$ be a graph. Let G' be obtained from G by subdividing every edge $e \in E$ into an odd number of pieces, by adding an even number m_e of new vertices. Let $m = \sum_e m_e$ be the total number of vertices added. Then G has a maximum independent set $I \subset V$ if and only if G' has a maximum independent set I' with $|I'| = |I| + m/2$.*

Proof. For every independent set $I \subseteq V$ in G , there is a corresponding independent set I' in G' with $|I'| = |I| + m/2$: for every pair of extra vertices on an edge, we can always add one of the two to an independent set. Conversely, for every independent set I' in G' , there is a corresponding independent set I in G with $|I| = |I'| - m/2$: I' cannot use both extra vertices on an edge, so if we simply remove all extra vertices we remove at most $|E|$ elements from I' (clearly, if we remove less than $m/2$ vertices from I' this way, we can remove more vertices until I has the desired cardinality). \square

Corollary 2. *Maximum independent set is also NP-hard on tripartite graphs.*

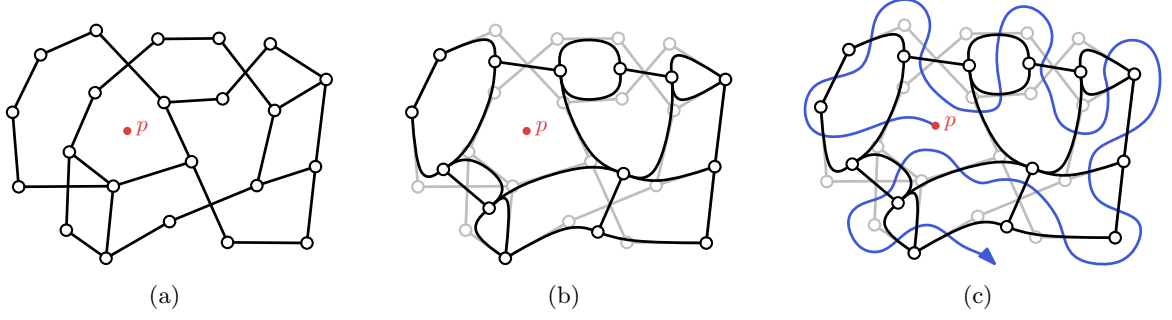


Figure 3: (a) An oddly embedded graph with four crossings. (b) The crossings are flattened according to the parity of their vertices. (c) An odd Euler path from p to the outer face.

Corollary 3. *Maximum independent set is NP-hard on oddly embeddable graphs.*

Given an embedded graph Γ , we say that a curve in the plane is an *odd Euler path* if it does not go through any vertex of Γ and it crosses every edge of Γ an odd number of times.

Lemma 3. *Let p be a point in the plane, and Γ an oddly embedded graph around p . Then there exists an odd Euler path for Γ that starts at p and ends in the outer face. Moreover, such path can be computed in polynomial time.*

Note that in particular, if Γ is a plane graph, then the lemma follows directly from the conventional notion of Euler paths in the dual (multi-)graph.

Proof. First, we insert an even number of extra vertices on every edge of Γ such that in the resulting embedded graph Γ' , every edge crosses at most one other edge. Now we construct an Euler path that crosses every edge of Γ' exactly once; note that this path will therefore cross every edge of Γ an odd number of times. Consider a pair of crossing edges and the four vertices concerned. For each pair of consecutive vertices (vertices that are not endpoints of the same edge), find a path in the graph that does not go around p (when seen as a cycle, after adding the crossing).

The parity of the length of this path does not depend on which path we take: if there would be an even-length path and an odd-length path between the same two vertices, both of which do not go around p , then there would be an odd cycle that does not contain p , which contradicts the oddly embeddedness of Γ' . Now, if the path has even length, we identify these two vertices. Note that of the four pairs of vertices involved in a crossing (i.e. ignoring the two pairs forming edges in Γ), exactly two pairs will have odd length connecting paths, so effectively we “flatten” the crossing. We do this for all crossings, and call the resulting multigraph Γ'' . See Fig. 3(b). (If the two crossing edges belong to different connected components of Γ , there are no paths connecting their vertices; in this case we make an arbitrary choice of which vertices to identify.)

Now Γ'' is planar. Furthermore, by construction, all faces of Γ'' have even length, except the one containing p and the outer face. Therefore, the dual multigraph of Γ'' has only two vertices of odd degree, and hence has an Euler path between these vertices. Furthermore, this Euler path crosses every edge of Γ' exactly once, and therefore every edge of Γ an odd number of times. Notice that the proof is constructive. Moreover, both the transformations and the Euler path can be done in polynomial time, hence such path can also be obtained in polynomial time. \square

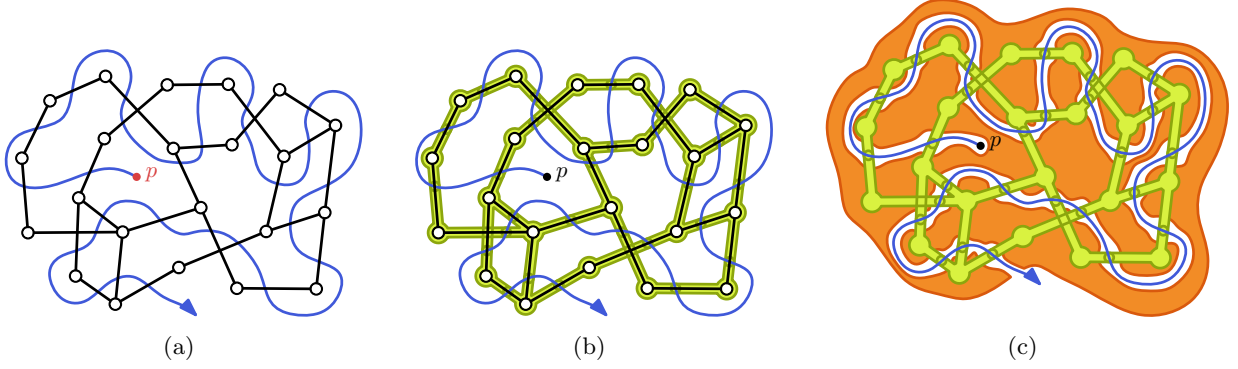


Figure 4: Creating regions to follow Γ and T .

3.2 Reduction from resilience to maximum independent set

Lemma 4. *Let p be a given point in the plane, and Γ an oddly embedded graph (not necessarily planar) around p . Furthermore, let T be a curve that forms an odd Euler path from p to the outer face. Then we can construct a set \mathcal{D} of connected regions such that a minimum set of regions from \mathcal{D} to remove corresponds exactly to a minimum vertex cover in Γ .*

Proof. If T is self-intersecting, we can rearrange the pieces between self-intersections to remove all self-intersections. Thus we assume that T is a simple path.

If T crosses any edge of Γ more than once, we insert an even number of extra vertices on that edge such that afterwards, every edge is crossed exactly once. Since we inserted an even number of vertices on every edge, by Observation 1 this does not affect the maximum independent set. Let Γ' be the resulting graph.

Now, for each vertex v in Γ' , we create one region D_v in \mathcal{D} . This region consists of the point where v is embedded, and the pieces of the edges adjacent to v up to the point where they cross T . Fig. 4(b) shows an example (the regions have been dilated by a small amount for visibility; if the embedding Γ has enough room this does not interfere with the construction). Note that all regions are simply connected.

Finally, we create one more special region W in \mathcal{D} that forms a corridor for T . Then W is duplicated at least n times to ensure that crossing this “wall” will always be more expensive than any other solution. Fig. 4(c) shows this.

Now, in order to escape, anyone starting at p must roughly follow T in order to not cross the wall. This means that for every edge of Γ' that T passes, one of the regions blocking the path (one of the vertices incident to the edge) must be disabled. The smallest number of regions to disable to achieve this corresponds to a minimum vertex cover in Γ' . \square

Recall that minimum vertex cover (or maximum independent set) is NP-hard on oddly embedded graphs by Corollary 3. As a consequence, we obtain our first hardness result for the barrier resilience problem (but note that this also follows from [1] and [14], although their constructions are quite different from ours).

Theorem 1. *The barrier resilience problem for arbitrary connected regions in the plane is NP-hard.*

Using fat regions. We now adapt the approach to also work for a much more restricted class of regions: axis-aligned rectangles of sizes $1 \times (1 + \varepsilon)$ and $(1 + \varepsilon) \times 1$ for any $\varepsilon > 0$ (as long as

ε depends polynomially on n). For simplicity, we limit Γ to have maximum degree 3. Maximum independent set is still known to be NP-hard in that case [6], and making them tripartite does not change the maximum degree.

The idea of the reduction is the following. We start from a sufficiently spacious (but polynomial) embedding of Γ , as illustrated in Fig. 5(a). On each edge we add a large even number of extra vertices. Each vertex will be replaced by a rectangle, so every edge in Γ will become a chain of overlapping rectangles, like the green rectangles in Fig. 5(b). Therefore the first phase consists in replacing the embedding of Γ by an equivalent embedding of rectangles. We call these rectangles *graph rectangles* (green in the figures). Some care must be taken in the placement of graph rectangles around degree-3 vertices and in crossings, so that the rest of the construction can be made to work. Next, we place *wall rectangles* (orange in the figures; these consist of many copies of the same rectangle) across each graph rectangle. The gaps between adjacent wall rectangles should cover the overlapping part of two adjacent graph rectangles, so that a path can pass through them only whenever one of the two graph rectangles is removed. Then, we find a curve T from p that goes through every gap exactly once (note that T exists, by Lemma 3). Fig. 5(c) illustrates this phase of the construction. Finally, we add more wall rectangles around T , to force any potential escape path from p that doesn't go through the wall rectangles to be homotopic to T . Fig. 5(d) shows the final set of rectangles. Now, computing an optimal resilience path among this set of rectangles would correspond to a maximal independent set in Γ .

For the construction to work, there needs to be enough space to place the wall rectangles. It is clear that this is possible far away from the graph rectangles, but close to the graph rectangles we proceed as follows: first, Fig. 6(a) shows the placement of rectangles along an edge of Γ . Fig. 6(b) shows how to place the rectangles at degree-3 vertices. Crossings are handled as shown in Fig. 6(c). These gadgets force some of the gaps in the chain to join each other. But this is no problem if every edge has enough rectangles. Also, note that at the center of the construction in Fig. 6(c) there are two overlapping green rectangles, which belong to the two crossing chains. This is the only place where we vitally use the fact that the regions are not pseudodisks.

Lemma 5. *Let p be a given point in the plane, and Γ an oddly embedded graph with maximum vertex degree 3 (not necessarily planar) around p . Furthermore, let T be a curve that forms an odd Euler path from p to infinity. Then we can construct a set \mathcal{D} of axis-aligned rectangles of aspect ratio $1 : (1 + \varepsilon)$ such that a minimum set of regions from \mathcal{D} to remove corresponds exactly to a minimum vertex cover in Γ .*

Proof. We first add groups of extra vertices on every edge of Γ so that we have room to place the rectangles, in an even number per edge. Then replace edges by chains as of rectangles as in Fig. 6, and connect the orange (wall) rectangles to force the only optimal path from p to the outer face to be along the Euler path T . The path may have to be rerouted locally close to the crossings, but since there is a sufficiently large number of crossings with every edge anyway, this is always possible. Orange rectangles have to be duplicated sufficiently many times again, to make sure that no optimal path will ever cross them. \square

Theorem 2. *The barrier resilience problem for regions that are axis-aligned rectangles of aspect ratio $1 : (1 + \varepsilon)$ is NP-hard.*

A similar approach can likely be used to show NP-hardness for other specific shapes of regions. However, it seems that a vital property is that they need to be able to completely cross each other: that is, the regions in \mathcal{D} should not be pseudodisks. Thus, if one were to prove that vertex cover for oddly embeddable graphs of bounded degree is NP-hard would also imply that the barrier resilience problem for unit disks is also NP-hard.

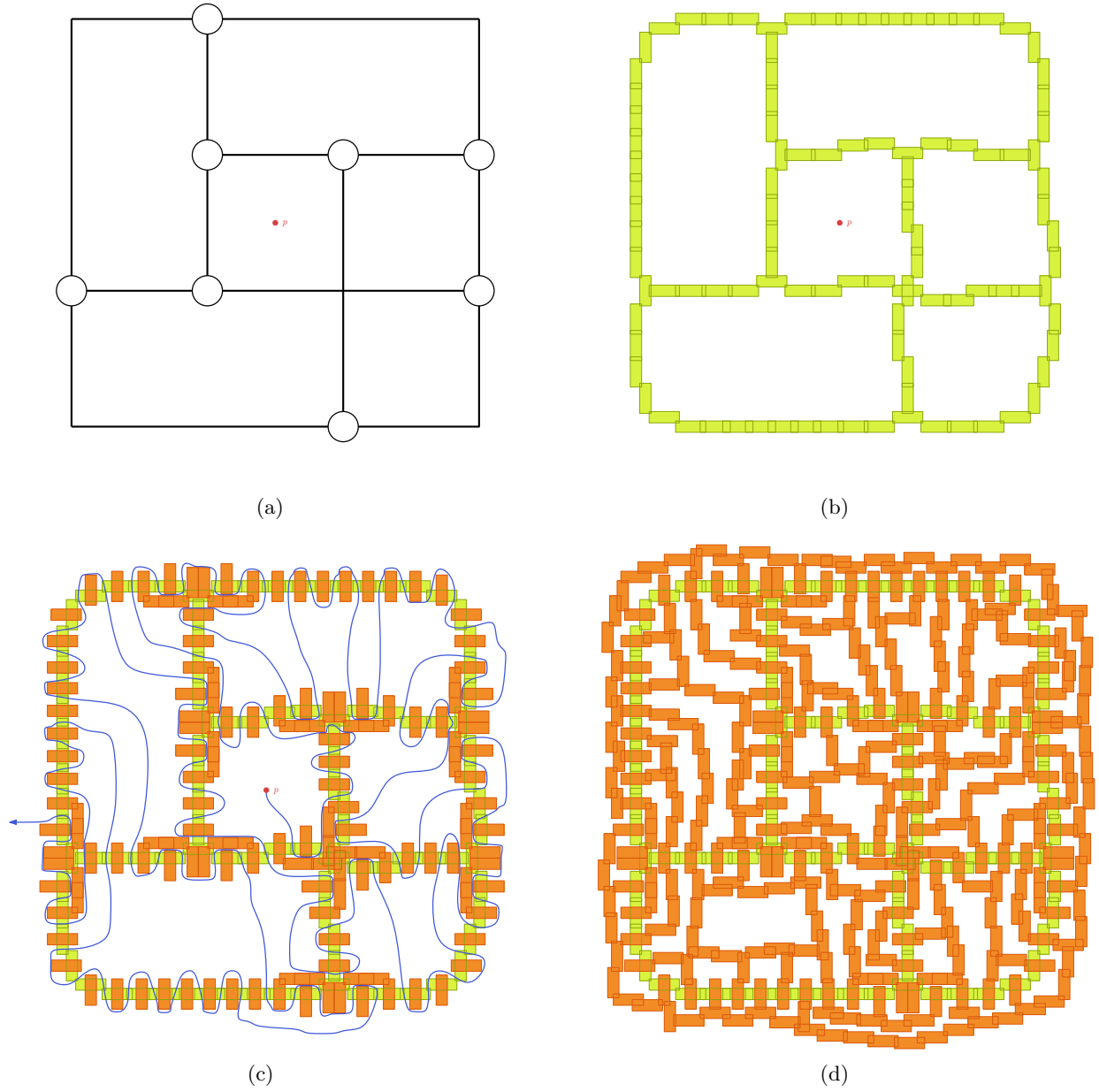


Figure 5: (a) A non-planar oddly embedded cubic graph, embedded on a grid. (b) A set of rectangles, containing exactly one rectangle for each input vertex, and an even number of rectangles for each input edge. Note that crossings can be embedded if sufficiently far apart. (c) Local walls are added to make “tunnels”, each tunnel contains the overlapping part of two adjacent yellow rectangles. To go through a tunnel, one of the two yellow rectangles has to be removed. Then we choose an Euler path from p to the outside, that goes through each tunnel exactly once. (d) The final set of rectangles, designed to force any path from p to the outside to be homotopically equal to the one we drew.

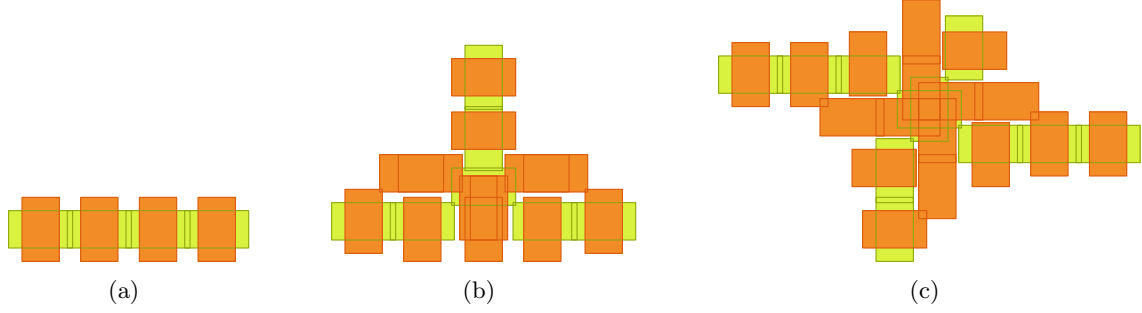


Figure 6: Local details of the construction. Note that we use rectangles with a large aspect ratio for visibility, but the same constructions can be made with aspect ratio arbitrarily close to 1. (a) Overlapping rectangles to create edges (with an even number of extra vertices). (b) A vertex of degree at most 3, which is just a single rectangle. (c) A crossing between two chains.

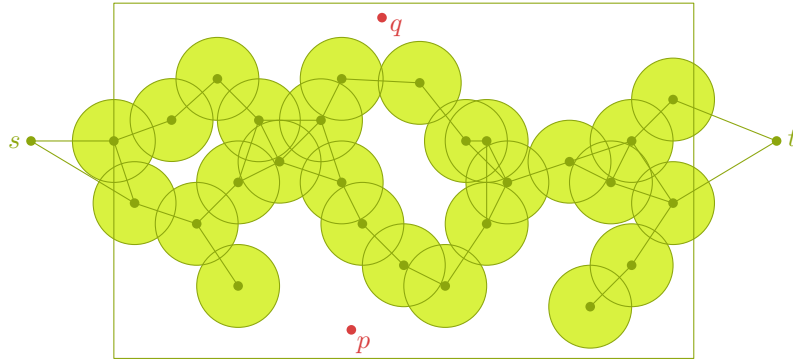


Figure 7: Any minimum vertex cut disconnecting s and t gives a minimum resilience path between p and q (that stays in the rectangular region).

4 Fixed-parameter tractability

In this section we introduce a fixed-parameter tractable (FPT) algorithm for the case in which \mathcal{D} is a collection of β -fat objects. For clarity we first explain the algorithm for the special case of unit disks. In Section 4.2 we show how to adapt the solution to fat regions. The parameter of our FPT algorithm is the length of the optimal solution, thus our aim is to obtain an algorithm that, given a problem instance whose solution has resilience r , runs in $O(2^{f(r)}n^c)$ time, for some constant c and some polynomial function f . Note that for treating the case of unit disk regions we assume that p and q are well-separated, so we can apply Lemma 1.

First we give a quick overview of the method of Kumar *et al.* [10] for open belt regions. Their idea consists in considering the intersection graph of \mathcal{D} together with two additional artificial vertices s, t with some predefined adjacencies. There is a path from the bottom side to the top side of the belt if and only if there is no path between s and t in the graph. Hence, computing the resilience of the network is equal to finding a minimum cut between s and t (see Fig. 7). Our approach is to find for a long path that passes through p and q , cut open and transform the problem instance into one with something similar to an open belt region. We then follow the approach of Kumar *et al.* taking into account that the right and left boundaries of our region correspond to the same point. Hence, instead of using a regular vertex cut, we will need to use a vertex multicut [15].

Consider the shortest path τ between the cells containing p and q in $G_{\mathcal{A}(\mathcal{D})}$, let t be the number

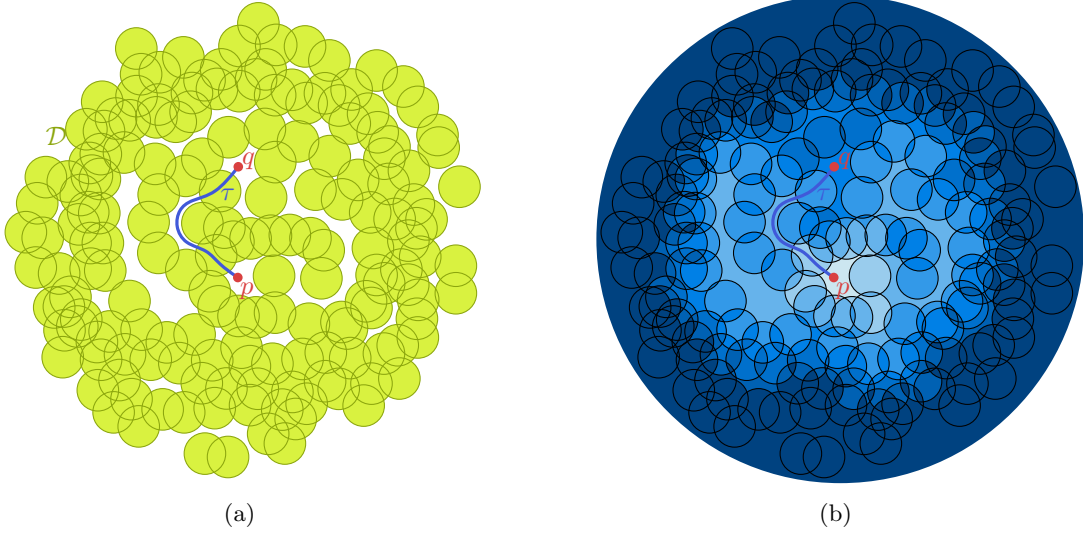


Figure 8: (a) A set of regions \mathcal{D} , and a shortest (minimum thickness) path between p and q . (b) The arrangement shaded by thickness from p .

of traversed disks (recall that we assumed that p is not contained in any region, hence this number is exactly the *thickness* of p and q , see Fig. 8(a)). We observe that cells with high thickness can be ignored when we look for low resilience paths.

Lemma 6. *The minimum resilience path between p and q cannot traverse cells whose thickness to p or q is larger than $1.5t$.*

Proof. We argue about thickness to p ; the argument with respect to q is analogous. Let ρ be a path of minimum resilience between p and q , and let r be the resilience of ρ . Recall that ρ does not enter a disk more than twice, hence the thickness of ρ is at most $2r \leq 2t$. Assume, for the sake of contradiction, that the thickness of some cell C traversed by ρ is greater than $1.5t$. Let ρ_C be the portion of ρ from C to q . By the triangle inequality, the thickness of ρ_C is less than $0.5t$. However, by concatenating τ and ρ_C we would obtain a path that connects p with C whose thickness is less than $1.5t$, giving a contradiction. \square

Let R be the union of the cells of the arrangement that have thickness from p at most $1.5t$; we also call R the *domain* of the problem. Fig. 9(a) shows an example. By the previous lemma, cells that do not belong to R can be discarded, since they will never belong to a path of resilience r , see Fig. 8(b).

Note that the number of cells remaining in R might still be quadratic, hence asymptotically speaking the instance size has not decreased (the purpose of this pruning will become clear later).

We extend the shortest path τ from q until a point q' on the boundary of the domain. Let τ' denote the extended path (Fig 9(b)).

Lemma 7. *There exists a path τ' from p to a point q' on the boundary of R via q , whose thickness is at most $1.5t$.*

Proof. Consider any shortest path tree from p in the dual graph of the reduced domain R , defined as the corresponding subgraph of $G_{\mathcal{A}(\mathcal{D})}$. All leaves of the tree correspond to cells on the boundary of R , which by definition have thickness at most $1.5t$ from p . Therefore all other cells in the tree lie on a path from p to a boundary cell that has length exactly $1.5t$, including q . \square

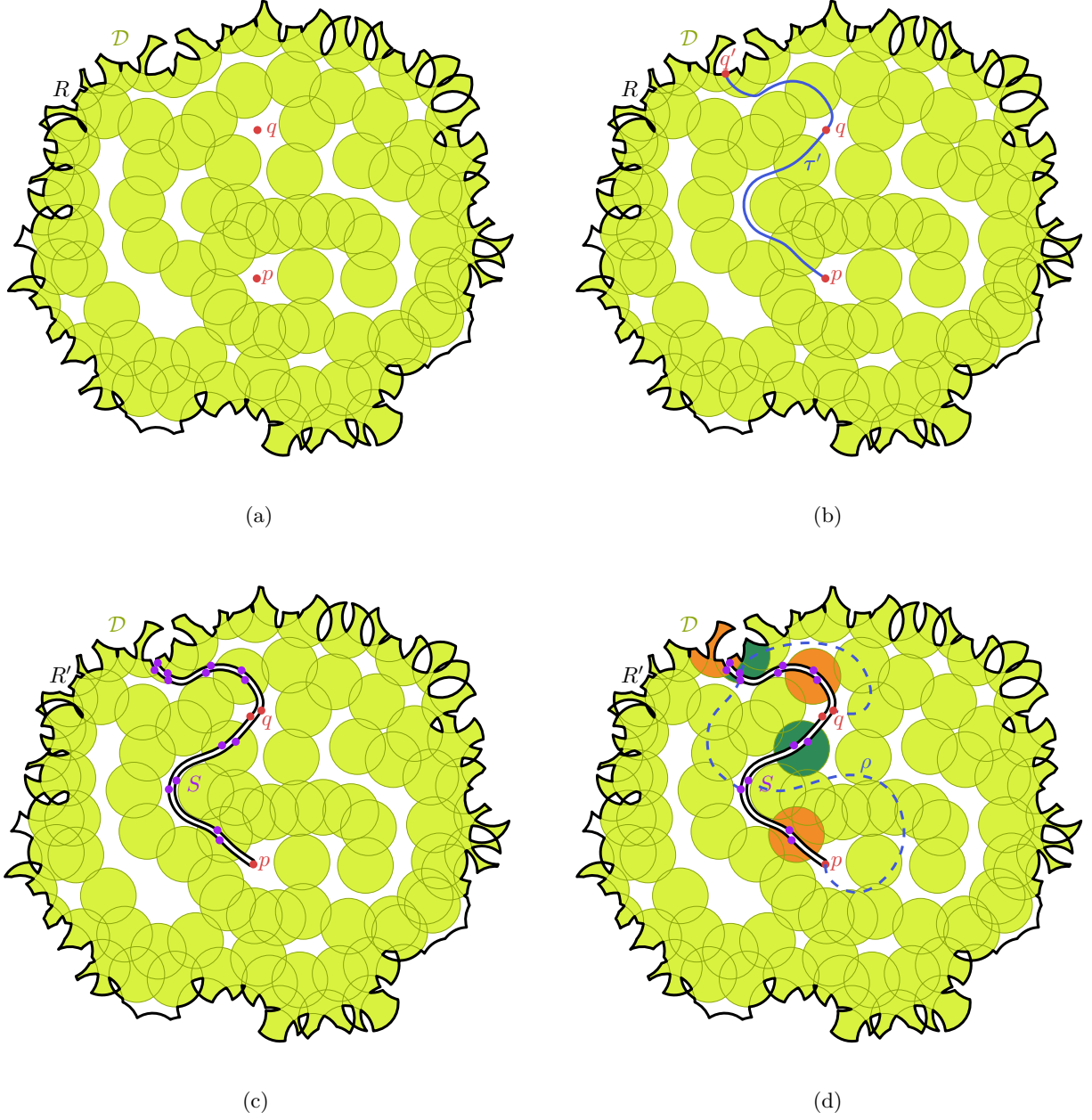


Figure 9: (a) The reduced domain R . (b) A path τ' from p to q' via q . (c) After cutting along τ' , we get the domain R' . We add a set S of extra vertices on the boundary of R' , and we now have two copies of q . (d) A crossing pattern, consisting of a topological path ρ (defined by the sequence of points of S it passes), and a binary assignment to the disks of \mathcal{D} intersected by τ' .

We “cut open” through τ' , removing the cut region from our domain. Note that cells that are traversed by τ' are split by two copies of the same Jordan curve (Fig 9(c)). After this cut we have two paths from p to q . We arbitrarily call them the left and right paths.

Consider now a minimum resilience path denoted ρ ; let $r = r(\rho)$ denote its resilience. This path can cross τ' several times, and it can even coincide with τ' in some parts (shared subpaths). Although we do not know how and where these crossings occur, we can *guess* (i.e., try all possibilities) the topology of ρ with respect to τ' . For each disk that τ' passes through, we either remove it (at a cost of 1) or we make it an obstacle. That way we explicitly know which of the regions traversed by τ' could be traversed by ρ . Additionally, we guess how many times ρ and τ' share part of their paths (either for a single crossing in one cell, or for a longer shared subpath). For each shared subpath, we guess from which cell ρ arrives and leaves (and if the entry or exit was from the left or right path). We call each such configuration a *crossing pattern* between τ' and ρ . Fig. 9(d) illustrates a crossing pattern.

Lemma 8. *For any problem instance \mathcal{D} , there are at most $2^{4r \log r + o(r \log r)}$ crossing patterns between τ' and ρ , where $r = r(\rho)$.*

Proof. First, for all disks in τ' , we guess whether or not they are also traversed by ρ . By Lemma 7, τ' has thickness at most $1.5t$, there are at most such many disks (hence up to $2^{1.5t}$ choices for which disks are traversed by ρ).

We now bound the number of (maximal) shared subpaths between ρ and τ' : recall that ρ passes through exactly $r = r(\rho)$ disks, and visits each disk at most twice. Hence, there cannot be more than $2r$ shared subpaths. Observe that τ' cannot traverse many cells of $\mathcal{A}(\mathcal{D})$: when moving from a cell to an adjacent one, we either enter or leave a disk of \mathcal{D} . Since we cannot leave a disk we have not entered and τ' has thickness at most $1.5t$, we conclude that at most $3t$ cells will be traversed by τ' (other than the starting and ending cells).

For each shared subpath we must pick two of the cells traversed in τ' (as candidates for first and last cell in the subpath). By the previous observation there are at most $3t$ candidates for first and last cell (since that is the number of cells traversed by τ'). Additionally, for each shared subpath we must determine from which side ρ entered and left the subpath (four options in total). Since these choices are independent, in total we have at most $2r \times (3t \times 3t \times 4)^{2r} = 2r \cdot 36^{2r} \cdot t^{4r}$ options for the number of crossing patterns. Combining both bounds and using the fact that $t \leq 2r$, we obtain:

$$2^{1.5t} \times 2r \cdot 36^{2r} \cdot t^{4r} \leq 2^{5r} \times 2r \cdot 36^{2r} \cdot (2r)^{4r} = 2^{9r+1+\log r+2r \log 36+4r \log r} = 2^{4r \log r + o(r \log r)}$$

□

Notice that the bound is very loose, since most of the choices will lead to an invalid crossing pattern (for example, patterns that forbid passing through a disk, but force ρ to pass through one of its cells). However, the importance of the previous lemma is in the fact that the total number of crossing patterns only depends on r (and not on n).

Our FPT algorithm consists in considering all possible crossing patterns, finding the optimal solution for a fixed crossing pattern, and returning the solution of smallest resilience. From now on, we assume that a given pattern has been fixed, and we want to obtain the path of smallest resilience that satisfies the given pattern. If no path exists, we simply discard it and associate infinite resilience to it.

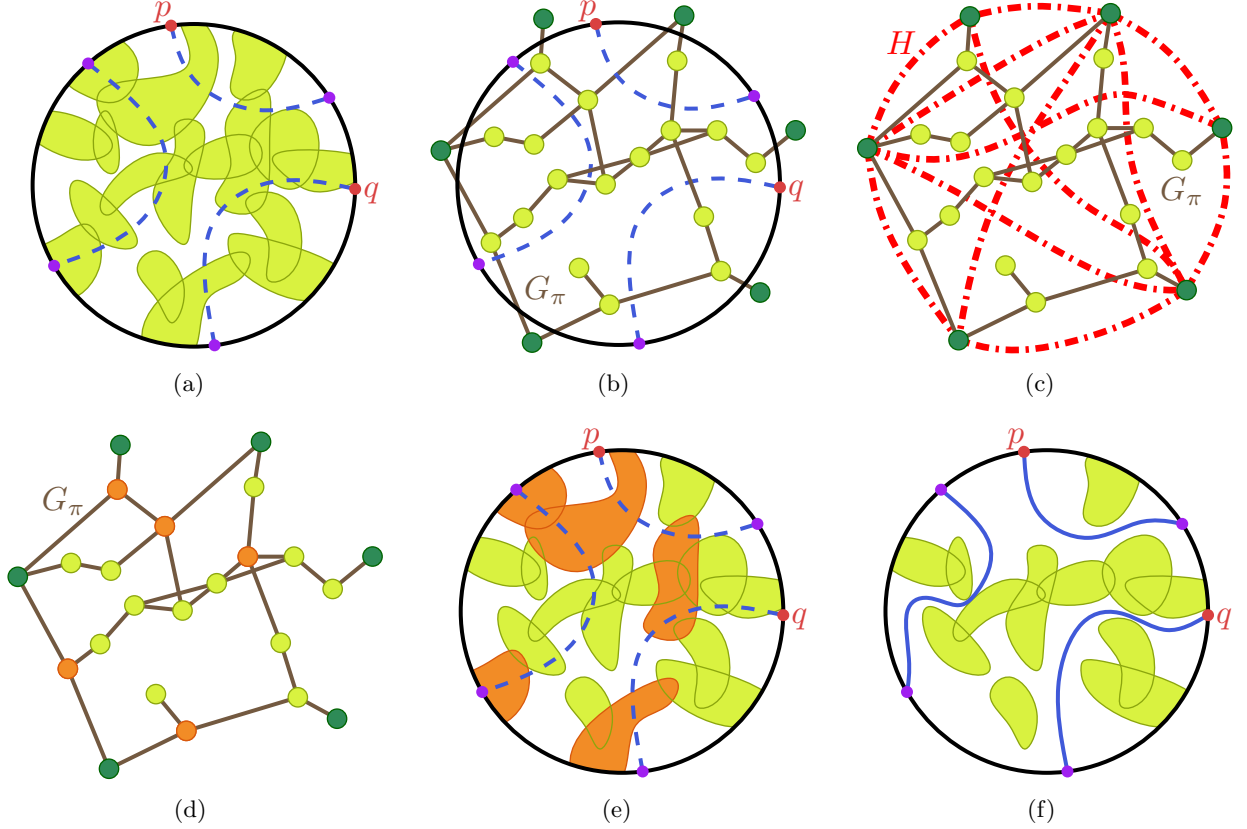


Figure 10: (a) We may schematically represent W as a circle, since the geometry no longer plays a role. Partial paths are shown dashed. (b) The intersection graph of the regions after adding extra vertices for boundary pieces between points of $S \cup \{p, q\}$, shown green. (c) The secondary graph H , representing the forbidden pairs. (d) A possible solution (the edges to remove are marked in orange). (e) The multicut vertices, translated back to the original regions. (f) The final set of regions and a path of resilience zero from p to q .

4.1 Solving the problem for a fixed crossing pattern

Recall that the crossing pattern gives us information on how to deal with the disks traversed by τ' . Thus, we remove all cells of the arrangement that contain one or more disks that are forbidden to ρ . Similarly, we remove from \mathcal{D} the disks that ρ must cross. After this removal, several cells of our domain may be merged.

Since we do not use the geometry, we may represent our domain by a disk W (possibly with holes). After the transformation, each remaining region of \mathcal{D} becomes a pseudodisk, and ρ becomes a collection of disjoint partial paths, each of which has its endpoints on the boundary of W (see Fig. 10(a)). To solve the subproblem associated with the crossing pattern we must remove the minimum number of disks so that all partial paths are feasible.

We consider the intersection graph G_I between the remaining regions of \mathcal{D} . That is, each vertex represents a region of \mathcal{D} , and two vertices are adjacent if and only if their corresponding regions intersect. Similarly to [10], we must augment the graph with boundary vertices. The partial paths split the boundary of R into several components. We add a vertex for each component (these vertices are called *boundary vertices*). We connect each such vertex to vertices corresponding to pseudodisks that are adjacent to that piece of boundary (Fig. 10(b)). Let $G_{\mathcal{X}} = (V_{\mathcal{X}}, E_{\mathcal{X}})$ be the

resulting graph associated to crossing pattern \mathcal{X} . Note that no two boundary vertices are adjacent.

We now create a secondary graph H as follows: the vertices of H are the boundary vertices of $G_{\mathcal{X}}$. We add an edge between two vertices if there is a partial path that separates the vertices in $G_{\mathcal{X}}$ (Fig. 10(c)). Two vertices connected by an edge of H are said to form a *forbidden pair* (each partial path that would create the edge is called a *witness* partial path). We first give a bound on the number of forbidden pairs that H can have.

Lemma 9. *Any crossing pattern has at most $2r^2 + r$ forbidden pairs.*

Proof. First notice that there are only “a few” boundary vertices: since partial paths cannot cross, each such path creates a single cut of the domain. This cut introduces a single additional boundary vertex (except the first partial path that introduces two vertices). Recall that we can map the partial paths to crossings between paths τ' and ρ and, by Lemma 8, these paths can cross at most $2r$ times. In particular, there cannot be more than $2r + 1$ boundary vertices. Observe that we only add a forbidden pair between boundary vertices of $G_{\mathcal{X}}$, hence in the worst case we have at most $\binom{2r+1}{2} = 2r^2 + r$ forbidden pairs. \square

The following lemma shows the relationship between the vertex multicut problem and the minimum resilience path for a fixed pattern.

Lemma 10. *There are k vertices of $G_{\mathcal{X}}$ whose removal disconnects all forbidden pairs if and only if there are k disks in \mathcal{D} whose removal creates a path between p and q that obeys the crossing pattern \mathcal{X} .*

Proof. Let \mathcal{A}' be the regions of $\mathcal{A}(\mathcal{D})$ inside R that are not covered by any disk (after the k disks have been removed), and let R' be their union. Two points are connected if and only if they belong to the same region of \mathcal{A}' . Hence, there is a path between p and q with the fixed crossing pattern if all partial paths are inside R' . The reasoning is analogous to the one used by Kumar *et al.* [10]. If all partial paths are possible, then no forbidden pair can remain, since—by definition—each forbidden pair disconnects at least one partial path (the witness path). On the other hand, as soon as one forbidden pair remains connected, there must exist at least one partial path (the witness path) that crosses the forbidden pair. Thus if a forbidden path is not disconnected, there can be no path connecting p and q . Fig. 10(e) illustrates how the resulting set of vertices corresponds to the original question. \square

That is, thanks to Lemma 10, we can transform the barrier resilience problem to the following one: given two graphs $G = (V, E)$, and $H = (V, E')$ on the same vertex set, find a set $D \subset V$ of minimum size so that no pair $(u, v) \in E'$ is connected in $G \setminus D$. This problem is known as the (vertex) *multicut* problem [15]. Although the problem is known to be NP-hard if $|E'| > 2$ [9], there exist several FPT algorithms on the size of the cut and on the size of the set E' [12, 15]. Among others, we distinguish the method of Xiao ([15], Theorem 5) that solves the vertex multicut problem in roughly $O((2k)^{k+\ell/2}n^3)$ time, where k is the number of vertices to delete, $\ell = |E'|$, and n is the number of vertices of G .

Theorem 3. *Let \mathcal{D} be a collection of unit disks in \mathbb{R}^2 , and let p and q be two well-separated points. There exists an algorithm to test whether $r(p, q) \leq r$, for any value r , and if so, to compute a path with that resilience, in $O(2^{f(r)}n^3)$ time, where $f(r) = r^2 \log r + o(r^2 \log r)$.*

Proof. Recall that our algorithm considers all possible crossings between ρ and τ' . For any fixed crossing pattern \mathcal{X} , our algorithm computes $G_{\mathcal{X}}$, and all associated forbidden pairs. We then execute Xiao’s FPT algorithm [15] for solving the vertex multicut problem. By Lemma 10, the

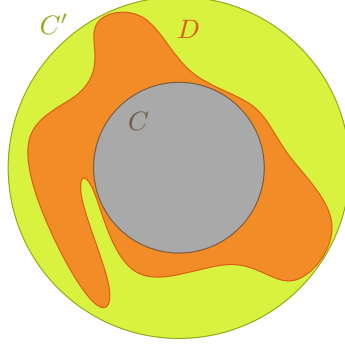


Figure 11: A β -fat region D is contained in a big disk, but contains a smaller disk; in this example, $\beta = 2$.

number of removed vertices (plus the number of disks that were forced to be deleted by \mathcal{X}) will give the minimum resilience associated with \mathcal{X} . Note that the disks to be deleted can be retrieved as well.

We now focus on the running time. Computationally speaking, the most expensive part of the algorithm is running an instance of the vertex multicut problem for each possible crossing pattern. Observe that the parameters k and ℓ of the vertex multicut problem are bounded by functions of r as follows: $k \leq r$ and $\ell \leq 2r^2 + r$ (the first claim is direct from the definition of resilience, and the second one follows from Lemma 9). Hence, a single instance of the vertex multicut problem will need $O((2r)^{r+(2r^2+r)/2} n^3) = O(2^{(1+\log r)(r^2+1.5r)} n^3) = O(2^{r^2 \log r + o(r^2 \log r)} n^3)$ time. By Lemma 8 the number of crossing patterns is bounded by $2^{4r \log r + o(r \log r)}$. Thus, by multiplying both expressions we obtain the bound on the running time, and the theorem is shown. \square

We remark that the main focus of this research is the fact that an FPT algorithm exists. Hence, although the dependency on r is high, we emphasize that the bounds are rather loose.

4.2 Extension to Fat Regions

We now generalize the algorithm to similarly-sized β -fat regions. A region D is β -fat if there exist two concentric disks C and C' whose radii differ by at most a factor β , such that $C \subseteq D \subseteq C'$ (whenever the constant β is not important, the region D is simply called *fat*). Since we need the regions to be of similar size, we assume without loss of generality that the radius of C is 1 and the radius of C' is β ; in this case we will call D a β -fat unit region. Fig. 11 shows an example of a 2-fat region. We first need to extend Lemma 1.

Lemma 11. *Let \mathcal{D} be a set of β -fat unit regions, and let $S \subset \mathcal{D}$ be an optimal solution. In the sequence of disks of S found when going from p to q (in an optimal way), no disk of S appears more than $(2\beta + 1)^2$ times.*

Proof. Let D be a region in S , and consider its containing disk C' , with center c . Clearly, the number of times an optimal path visits D is upper-bounded by the number of times it visits C' . Now, analogously to the original argument by Bereg and Kirkpatrick [2], every time the optimal path visits and leaves C' , it must do so in order to avoid some other region. This other region must intersect D , and since it is β -(unit)fat, it must contain a unit disk centered at distance at most β from D . Therefore all regions intersecting D have their unit-disks centered at distance at most 2β from c . Moreover, they are totally contained in a disk of radius $2\beta + 1$ centered at c . A simple area argument shows that at most $(2\beta + 1)^2$ disjoint unit-disks fit into a disk of radius $(2\beta + 1)$. \square

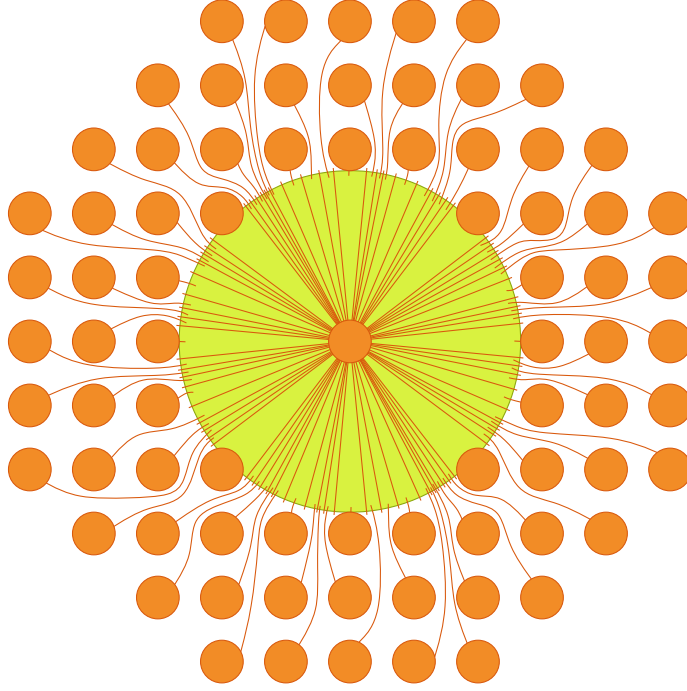


Figure 12: Example showing that the $\Theta(\beta^2)$ bound is tight.

Fig. 12 illustrates that a path may indeed visit the same region as many as $\Omega(\beta^2)$ times, so the bound in Lemma 11 is asymptotically tight.

Corollary 4. *When the regions of \mathcal{D} are β -fat unit regions, the thickness between two points is at most $(2\beta + 1)^2$ times their resilience.*

This change in the upper bound of the thickness in terms of the resilience implies similar changes in Lemmas 6, 7, 8 and 9. The following lemmas summarize these changes; they are proved in the same way as their counterparts for disks, thus we only sketch the differences with the original proofs (if any).

Lemma 12. *The minimum resilience path between p and q cannot traverse cells whose thickness to p or q is larger than $(1 + (2\beta + 1)^2)\frac{t}{2}$.*

Proof. (Sketch) We use the same reasoning as in the proof of Lemma 6. On the one hand there is the minimum thickness path between p and q , whose thickness is t . On the other hand, we also have the minimum resilience path ρ between the same points, whose thickness is at most $(2\beta + 1)^2 t$. Assume now that any cell C traversed by ρ has thickness $k(2\beta + 1)^2 t$ from p , for some $0 < k < 1$. The alternative path goes from p to C , via q , and has thickness $(1 - k)(2\beta + 1)^2 t + t$. The bound we need is obtained for the value of k that makes both expressions equal, which is $k = \frac{1}{2} + \frac{1}{2(2\beta + 1)^2}$, leading to the claimed value. \square

Lemma 13. *There exists a path τ' from p to a point q' on the boundary of R via q , whose thickness is at most $(1 + (2\beta + 1)^2)\frac{t}{2}$.*

Lemma 14. *For any problem instance \mathcal{D} , there are at most $2^{O(\beta^4 r + \beta^2 r \log(\beta r))}$ crossing patterns between τ' and ρ .*

Proof. (Sketch) Let $\mu = (2\beta + 1)^2$ and $\nu = \frac{1+(2\beta+1)^2}{2}$. We proceed as in the proof of Lemma 8. Recall that previously we had $2^{1.5t} \times 2r \cdot (3t \times 3t \times 4)^{2r}$ crossing patterns, but now we must use the bounds that depend on β instead. What before was $2r$ now becomes μr , and the $1.5t$ terms now become νt (recall that $3t = 2 \cdot 1.5t$, hence they must be replaced as well). Making these changes in the previous expression, we obtain that the number of crossings is bounded by:

$$2^{\nu t} \times \mu r \cdot (2\nu t \times 2\nu t \times 4)^{\mu r}.$$

Using that $t \leq \mu r$, and simplifying the expression, we get:

$$\leq 2^{\nu \mu r} \times \mu r \cdot (4\nu \mu r)^{2\mu r} = 2^{\nu \mu r + \log(\mu r) + 2\mu r \log(4\nu \mu r)}$$

Finally, we apply that both $\mu, \nu \in O(\beta^2)$, and obtain the desired bound. \square

Lemma 15. *Any crossing pattern has at most $O(\beta^4 r^2)$ forbidden pairs.*

Proof. This follows from the fact that τ' and ρ traverse through at most $2\mu r$ cells of $\mathcal{A}(\mathcal{D})$ (where $\mu = (2\beta + 1)^2$), hence at most such many crossings can occur. Each crossing generates a single boundary vertex, hence the number of forbidden pairs is bounded by $O((2\mu r)^2) = O(\beta^4 r^2)$. \square

With these results in place, the rest of the algorithm remains unchanged: the only property of unit disks that is still used is the fact that they are connected, to be able to phrase the problem as a vertex cut in the region intersection graph.

Theorem 4. *Let \mathcal{D} be a collection of n connected β -fat unit regions in \mathbb{R}^2 , and let p and q be two points. Let r be a parameter. There exists an algorithm to test whether $r(p, q) \leq r$, and if so, to compute a path with that resilience, in $O(2^{f(\beta, r)} n^3)$ time, where $f(\beta, r) \in O(\beta^4 r^2 \log(\beta r))$.*

Proof. As before, the running time is bounded by the product of the number of crossing patterns and the time needed to solve a single instance of the vertex multicut problem. By Lemmas 14 and 15, these bounds now become $2^{O(\beta^4 r + \beta^2 r \log(\beta r))}$ and $2^{O(\beta^4 r^2 \log(\beta r))}$, respectively. The product of both is dominated by the second term, hence the theorem is shown. \square

5 $(1 + \varepsilon)$ -approximation

An arrangement \mathcal{D} is said to have bounded ply Δ if no point $p \in \mathbb{R}^2$ is contained in more than Δ elements of \mathcal{D} . In this section, we present a polynomial-time approximation scheme (PTAS) for computing the resilience of an arrangement of disks of bounded ply Δ . The general idea of the algorithm is very simple: first, we compute all pairs of regions that can be reached by removing at most k disks, where we will set $k = \lceil 4\Delta/\varepsilon^2 \rceil$; the reason for choosing this value will become clear later. Then, we compute the shortest path in the dual graph of the arrangement of regions, augmented with extra edges for these pairs. We prove that the length of the resulting path is a $(1 + \varepsilon)$ -approximation of the resilience. As in the previous section, we first consider the case in which \mathcal{D} is a set of n unit disks in \mathbb{R}^2 of ply Δ . In Section 5.4 we extend the result to β -fat regions.

Let $\mathcal{A}(\mathcal{D})$ be the arrangement induced by the regions of \mathcal{D} , and let $G_{\mathcal{A}(\mathcal{D})}$ be the dual graph of $\mathcal{A}(\mathcal{D})$. Recall that $G_{\mathcal{A}(\mathcal{D})}$ has a vertex for every cell of $\mathcal{A}(\mathcal{D})$, and a directed edge between all pairs of adjacent cells of cost 1 when entering a disk, and cost 0 when leaving a disk. For any given k , let G_k be the graph obtained from $G_{\mathcal{A}(\mathcal{D})}$ by adding, for each pair of cells $A, B \in \mathcal{A}(\mathcal{D})$ with resilience at most k , a *shortcut edge* \overrightarrow{AB} of cost $r(A, B)$.

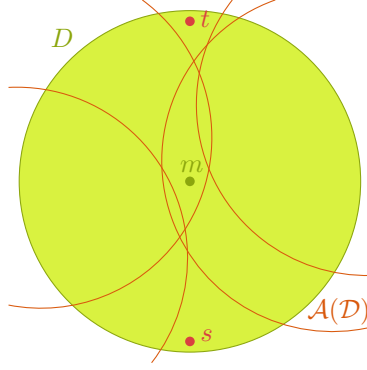


Figure 13: The point m is contained in at least $c' + 1$ disks of $\mathcal{A}(\mathcal{D})$. Therefore, one of the neighboring cells has ply at least $c' + c + 1$

For a pair of cells of $\mathcal{A}(\mathcal{D})$, we can test whether $r(A, B)$ is smaller than k , and if it is, compute it, in $O(2^{f(k)}n^3)$ time (where $f(k) = r^2 \log r + o(r^2 \log r)$) by applying Theorem 3 to a point $p \in A$ and a point $q \in B$. Since there are $O(n^2)$ cells in $\mathcal{A}(\mathcal{D})$, we can compute G_k by doing this $O(n^4)$ times, leading to a total running time of $O(2^{f(k)}n^7)$. Observe that this running time is polynomial in n , and exponential in k . In particular, it is a PTAS since $k = 4\Delta/\varepsilon^2$. Again, we emphasize that the bounds are loose, and that our objective is to show the existence of a PTAS to the resilience problem.

5.1 Analysis

We will now show that the algorithm described above yields a $(1 + \varepsilon)$ -approximation. We begin with the following observation.

Lemma 16. *Let $D \in \mathcal{D}$, where $\mathcal{A}(\mathcal{D})$ has ply Δ , and let s, t be any two points inside D . Then the resilience between s and t in \mathcal{D} is at most Δ .*

Proof. Let c be the number of disks containing either s or t ($c \geq 1$, since D contains both points). These c disks clearly must be removed. Now we analyze what other disks, not containing neither s nor t , may need to be removed too. For each other disk D_1 (not containing both s and t) that needs to be removed in an optimal solution, there must be another disk D_2 that intersects D_1 and, together, separate s and t inside D . We call such a pair of disks a *separating pair*.

Thus if the resilience is $(c + c')$, there must be at least c' *disjoint* separating pairs intersecting D . Moreover, since disks have unit-size, if two disks form a separating pair, at least one of them must intersect the center of D . Similarly, any disk containing s and t must contain the center of D as well. Fig. 13 illustrates the argument.

Since the ply of $\mathcal{A}(\mathcal{D})$ is Δ , this implies that there can be at most $\Delta - c$ separating pairs, and thus the resilience is at most Δ . \square

The previous lemma implies that in an optimal resilience path, if a disk appears twice, its two occurrences cannot be more than 2Δ apart (when counting the cells traversed by the path between the two occurrences of the disk).

To prove our result it will be convenient to focus at the sequence of disks encountered by a path when going from p to q . It turns out that such problem is essentially a string problem, where each symbol represents a disk encountered by the path. In that context, the thickness will be equivalent

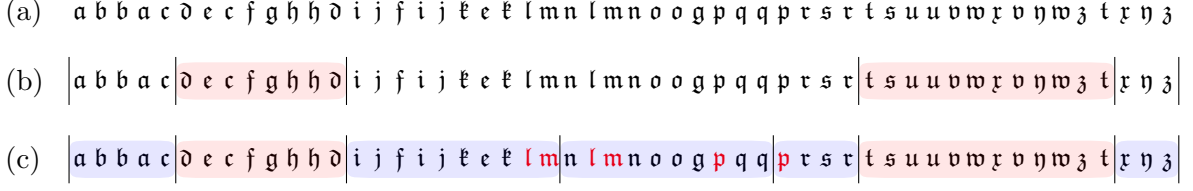


Figure 14: (a) A string of 52 symbols, each appearing twice. (b) First, we identify a maximal set of segments bounded by equal symbols, and longer than $\lambda = 4$. (c) Then, we segment the remaining pieces into segments of length $k = 10$. Red symbols are double-counted.

to the number of symbols of the string (recall that we assume that p is not contained in any disk), and the resilience to the number of distinct symbols.

5.2 Observations about strings

Let $S = \langle s_1 \dots s_n \rangle$ be a string of n symbols from some alphabet \mathfrak{A} , such that no symbol appears more than twice. Let T be a substring of S . We define $\ell(T)$ to be the length of T , and $d(T)$ to be the number of distinct symbols in T . Clearly, $\frac{1}{2}\ell(T) \leq d(T) \leq \ell(T)$. Let σ and k be two fixed integers such that $\sigma < k$. We define the *cost* of a substring T of S to be:

$$\psi(T) = \begin{cases} \sigma & \text{if } T = \langle a\lambda a \rangle \text{ for some } a \in \mathfrak{A}, \text{ string } \lambda \text{ such that } a \notin \lambda, \text{ and } \ell(T) > \sigma \\ d(T) & \text{if } \ell(T) \leq k \\ \ell(T) & \text{otherwise} \end{cases}$$

Note that, in the string context, d acts as the resilience, ℓ as the thickness, and ψ is the approximation we compute. Intuitively, if T is short (i.e., length at most k) we can compute the exact value $d(T)$. If T has a symbol whose two appearances are far away we will use a “shortcut” and pay σ (i.e., for unit disk regions, by Lemma 16, we will take $\sigma = \Delta$). Otherwise, we will approximate d by ℓ . Given a long string, we wish to subdivide S into a *segmentation* \mathcal{T} , composed of m disjoint segments (i.e. substrings of S) T_1, \dots, T_m , that minimize the total cost $\psi(\mathcal{T}) = \sum_i \psi(T_i)$. Clearly, $\psi(\mathcal{T}) \leq \ell(S)$.

Lemma 17. *Let S be a sequence. There exists a segmentation \mathcal{T} such that $\psi(\mathcal{T}) \leq (1 + \varepsilon)d(S)$, where $\varepsilon = 2\sqrt{\sigma/k}$.*

Proof. Let λ be an integer such that $\sigma < \lambda < k$, of exact value to be specified later. First, we consider all pairs of equal symbols in S that are more than λ apart. We would like to take all of these pairs as separate segments; however, we cannot take segments that are not disjoint. So, we greedily take the leftmost symbol s whose partner is more than λ further to the right, and mark this as a segment. We recurse on the substring remaining to the right of the rightmost s .^[2] Finally, we segment the remaining pieces greedily into pieces of length k . Fig. 5.2 illustrates the resulting segmentation.

Now, we prove that the resulting segmentation has a cost of at most $(1 + \varepsilon)d(S)$. First, consider a symbol to be *counted* if it appears in only one short (blue) segment, and to be *double-counted* if it appears in two different short segments. Suppose s is double-counted. Then the distance between its two occurrences must be smaller than λ , otherwise it would have formed a long (red) segment.

^[2]In fact, we could choose any disjoint collection such that after their removal there are no more segments of this type longer than λ .

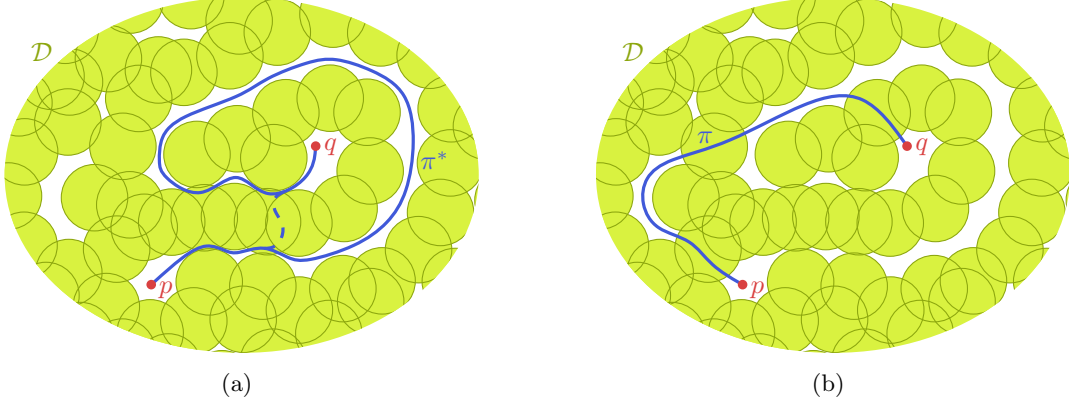


Figure 15: (a) The optimal path π^* , achieving a resilience of 2. There is a segmentation of π^* of cost 3, using the dashed shortcut. (b) A minimum cost path π found by the algorithm. In this example, the resilience of π is 3.

Therefore, it must appear in two adjacent short segments. The leftmost of these two segments has length exactly k , but only λ of these can have a partner in the next segment. So, at most a fraction λ/k symbols are double-counted.

Second, we need to analyze the cost of the long (red) segments. In the worst case, all symbols in the segment also appear in another place, where they were already counted. In this case, the true cost would be 0, and we pay σ too much. However, we can assign this cost to the at least λ symbols in the segment; since each symbol appears only twice they can be charged at most once. So, we charge at most σ/λ to each symbol. The total cost is then bounded by $(1 + \lambda/k + \sigma/\lambda)d(S)$. To optimize the approximation factor, we choose λ such that $\lambda/k = \sigma/\lambda$; more precisely we take $\lambda = \lceil \sqrt{k\sigma} \rceil$. Recall that we initially set $k = \lceil 4\sigma/\varepsilon^2 \rceil$. \square

5.3 Application to resilience approximation

We now show that the shortest path between any p, q in G_k is a $(1 + \varepsilon)$ -approximation of their resilience. Let π be a path from p to q in \mathbb{R}^2 , and let $S(\pi)$ be the sequence that records every disk of \mathcal{D} we enter along π , plus the disks that contain the start point of π , added at the beginning of the sequence, in any order. Then we have $|S(\pi)| = t(\pi)$.

Lemma 18. *For every path π from p to q and every segmentation \mathcal{T} of $S(\pi)$, there exists a path from p to q in G_k of cost at most $\psi(\mathcal{T})$.*

Proof. We describe how to construct a path in G_k based on \mathcal{T} . For every segment T of \mathcal{T} , we create a piece of path whose length in G_k is at most the cost of the segment $\psi(T)$.

There are three types of segments. The first type are segments that start and end with the same symbol \mathfrak{a} , which corresponds to a disk $D \in \mathcal{D}$. For those, we make a shortcut path that stays inside D , as per Lemma 16. The second type are segments whose length is at most k . For those, by definition, G_k contains a shortcut edge whose cost is exactly the resilience between the corresponding cells or $\mathcal{A}(\mathcal{D})$. The third type are the remaining segments. For those, we simply use the piece of π that corresponds to T . \square

Lemma 19. *For any points $p, q \in \mathbb{R}^2$, the resilience of $\mathfrak{P}_{G_k}(p, q)$ is at most $(1 + \varepsilon)r(p, q)$.*

Proof. Let π^* be a path from p to q of optimal resilience $r^* = r(\pi^*) = r(p, q)$. Then, consider the sequence $S(\pi^*)$, that is, the sequence of disks π^* enters. Now, by Lemma 17, there exists a segmentation \mathcal{T} of $S(\pi^*)$ of cost at most $(1 + \varepsilon)d(S(\pi^*)) = (1 + \varepsilon)r^*$. By Lemma 18, there exists a path in G_k of equal or smaller cost. Fig. 15(a) illustrates this.

Now, consider the path π our algorithm produces. The resilience of π is smaller than the cost of π in G_k , which is smaller than the cost of π^* in G_k , which is smaller than $1 + \varepsilon$ times the resilience of π^* . That is:

$$r(\pi) \leq \text{cost}_{G_k}(\pi) \leq \text{cost}_{G_k}(\pi^*) \leq (1 + \varepsilon)r(\pi^*) = (1 + \varepsilon)r^*$$

□

We conclude with our main result in this section.

Theorem 5. *Let \mathcal{D} be a set of unit disks of ply Δ in \mathbb{R}^2 . We can compute a path π between any two given points $p, q \in \mathbb{R}^2$ whose resilience is at most $\leq (1 + \varepsilon)r(p, q)$ in $O(2^{f(\Delta, \varepsilon)}n^7)$ time, where $f(\Delta, \varepsilon) = 16\frac{\Delta^2 \log(\Delta/\varepsilon)}{\varepsilon^4} + o(\frac{\Delta^2 \log(\Delta/\varepsilon)}{\varepsilon^4})$.*

Proof. The running time of the algorithm is dominated by the preprocessing stage: determining if the resilience between every pair of vertices of $G_{\mathcal{A}(\mathcal{D})}$ is at most $\lceil 4\Delta/\varepsilon^2 \rceil$. Since $G_{\mathcal{A}(\mathcal{D})}$ has $O(n^2)$ cells, in total we execute the algorithm of Theorem 3 at most $O(n^4)$ times, and we obtain the desired bound. □

5.4 Extension to Fat Regions

As in Section 4.2, we now generalize the result to arbitrary β -fat unit regions. For the purpose, we must extend Lemma 16.

Lemma 20. *Let $D \in \mathcal{D}$, where $\mathcal{A}(\mathcal{D})$ has ply Δ , and let p, q be any two points inside D . Then the resilience between p and q in \mathcal{D} is at most $(2\beta + 1)^2\Delta$.*

Proof. The resilience between p and q is upper-bounded by the number of regions that intersect D . We can give an upper bound using a simple packing argument. Since p and q belong to a β -(unit)fat region D , they are both inside a circle C with center c and radius β . Any other β -fat region D' that interferes with the path from p to q must intersect C . Such an intersecting region, being also β -fat, must contain a unit-disk whose center cannot be more than 2β away from c . Therefore all regions intersecting C have their unit-disks centered at distance at most β from c . Moreover, such disks are totally contained in a disk of radius $2\beta + 1$ centered at c . As in the proof of Lemma 11, we can show that at most $(2\beta + 1)^2$ disjoint unit-disks fit into a disk of radius $(2\beta + 1)$. Since the ply is at most Δ , the maximum number of unit-disks inside a disk of radius β in \mathcal{D} is $(2\beta + 1)^2\Delta$. □

As before, the rest of the arguments do not rely on the geometry of the regions anymore, and we can proceed as in the disk case. The only difference, is that the value σ of doing a shortcut has increased to $(2\beta + 1)^2\Delta$.

Theorem 6. *Let \mathcal{D} be a set of unit disks of ply Δ in \mathbb{R}^2 . We can compute a path π between any two points $p, q \in \mathbb{R}^2$ whose resilience is at most $\leq (1 + \varepsilon)r(p, q)$ in $O(2^{f(\Delta, \beta, \varepsilon)}n^7)$ time, where $f \in O(\frac{\Delta^2 \beta^6}{\varepsilon^4} \log(\beta\Delta/\varepsilon))$.*

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References

- [1] H. Alt, S. Cabello, P. Giannopoulos, and C. Knauer. On some connection problems in straight-line segment arrangements. In *Proc. EuroCG*, pages 27–30, 2011.
- [2] S. Bereg and D. G. Kirkpatrick. Approximating barrier resilience in wireless sensor networks. In *Proc. ALGOSENSORS*, pages 29–40, 2009.
- [3] C.-Y. Chang. The k-barrier coverage mechanism in wireless visual sensor networks. In *Proc. WCNC*, pages 2318–2322, 2012.
- [4] D. Z. Chen, Y. Gu, J. Li, and H. Wang. Algorithms on minimizing the maximum sensor movement for barrier coverage of a linear domain, 2012. Preprint. <http://arxiv.org/abs/1207.6409>.
- [5] M. de Berg, M. J. Katz, A. F. van der Stappen, and J. Vleugels. Realistic input models for geometric algorithms. In *Proc. SoCG*, pages 294–303, 1997.
- [6] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified np-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.
- [7] M. Gibson, G. Kanade, and K. Varadarajan. On isolating points using disks. In *Proc. ESA*, pages 61–69, 2011.
- [8] S. He, J. Chen, X. Li, X. Shen, and Y. Sun. Cost-effective barrier coverage by mobile sensor networks. In *Proc. INFOCOM*, pages 819–827, 2012.
- [9] T. C. Hu. Multi-commodity network flows. *Operations Research*, 11(3):pp. 344–360, 1963.
- [10] S. Kumar, T.-H. Lai, and A. Arora. Barrier coverage with wireless sensors. In *Proc. MOBICOM*, pages 284–298, 2005.
- [11] S. Kumar, T.-H. Lai, and A. Arora. Barrier coverage with wireless sensors. *Wireless Networks*, 13(6):817–834, 2007.
- [12] D. Marx. Parameterized graph separation problems. *Theoretical Computer Science*, 351(3):394–406, 2006.
- [13] G. Miller, S. Teng, W. Thurston, and S. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *Journal of the ACM*, 44(1):1–29, 1992.
- [14] K.-C. R. Tseng and D. Kirkpatrick. On barrier resilience of sensor networks. In *Proc. ALGOSENSORS*, pages 130–144, 2012.
- [15] M. Xiao. Simple and improved parameterized algorithms for multiterminal cuts. *Theory of Computing Systems*, 46(4):723–736, 2010.